Cumulative Review

Vectors

1. Show that the cross product of two unit vectors is not generally a unit vector.

2. Prove that $(\vec{u} \times \vec{v})$ is perpendicular to $\vec{v}$.

3. The points $A(2, 4)$, $B(0, 0)$, and $C(-2, 1)$ define a triangle in the plane. Find the cosine of $\angle ABC$.

4. Write the vector $(0, 8)$ as a linear combination of the vectors $(2, 4)$ and $(-2, 1)$.

5. For the four points $A(2k, 0, 0)$, $B(0, 2k, 0)$, $C(0, 0, 2k)$, and $D(2l, 2l, 2l)$, let W be the midpoint of AB, X the midpoint of BC, Y the midpoint of CD, and Z the midpoint of DA. Prove that W, X, Y, and Z are coplanar.

6. In ΔABC, P is the midpoint of BC. Q is the point that divides AP internally in the ratio 5:2. R is on AC such that $\overrightarrow{AR} = k\overrightarrow{AC}$, for k a real number. For what value of k is BQR a straight line?

7. $P_1, P_2, P_3, \ldots P_{12}$, are consecutive vertices of a regular polygon with 12 sides. If $\overrightarrow{P_1, P_2} = \vec{x}$ and $\overrightarrow{P_1P_3} = \vec{y}$, express the following vectors in terms of $\vec{x}$ and $\vec{y}$:
   a. $\overrightarrow{P_2P_3}$ b. $\overrightarrow{P_1P_4}$ c. $\overrightarrow{P_3P_7}$

8. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be linearly independent vectors in space, and let

$$\vec{u} = 3\vec{a} + 2\vec{b} - \vec{c}$$
$$\vec{v} = -2\vec{a} + 4\vec{c}$$
$$\vec{w} = -\vec{a} + 3\vec{b} + k\vec{c}$$

Determine k so that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are coplanar.

26. In the following system of equations, k is a real number.

$$-2x + y + z = k + 1$$
$$kx + z = 0$$
$$y \mp kz = 0$$

   a. For what value(s) of k does the system
      i) have no solution?
      ii) have exactly one solution?
      iii) have an infinite number of solutions?
   b. For part **a iii**, determine the solution set and give a geometric interpretation.

3. 0 **4.** $(0, 8) = \frac{8}{5}(2, 4) + \frac{8}{5}(-2, 1)$ **6.** $\frac{5}{9}$
7. a. $-\vec{x} + \vec{y}$ **b.** $(1 + \sqrt{3})\vec{y} - (1 + \sqrt{3})\vec{x}$
c. $(-6 - 4\sqrt{3})\vec{x} + (3 + 2\sqrt{3})\vec{y}$ **8.** $\frac{19}{2}$ **12. b.** $(-5, 5, 5)$
13. yes **14.** $2x - 3y - 3z - 12 = 0$ **15.** $A\left(\frac{7}{11}, \frac{4}{11}, 3\right)$, $B\left(\frac{3}{11}, \frac{6}{11}, \frac{27}{11}\right)$
16. $x + 2y + 2z - 20 = 0$ **17. a.** no intersection
b. $\vec{r} = (-1, 3, 0) + t(1, 2, -1)$ **c.** $\left(\frac{-5}{2}, 0, \frac{3}{2}\right)$ **18.** $(2, \frac{1}{2}, 0)$
19. $\left(2, \frac{5}{2}, \frac{5}{2}\right)$ **20. a.** $x = -1 - t, y = 3 + t, z = t$
b. xy plane at $(-1, 3, 0)$, xz plane at $(2, 0, -3)$,
yz plane at $(0, 2, -1)$ **c.** $3\sqrt{3}$ **21.** $\left(\frac{103}{11}, \frac{-93}{11}, \frac{60}{11}\right)$
22. $(24, 36, 8)$ **23.** $\pm \frac{3}{2\sqrt{109}}$ **24.** $a = \frac{1}{2}b, b \neq -2$
25. $(x, y, z) = (3, -1, 0)$ **26. a. (i)** $k = 2$ **(ii)** $k \neq 2, k \neq -1$
(iii) $k = -1$ **b.** planes intersect in the line $(x, y, z) = (t, t, t)$

9. Prove that the diagonals of a parallelogram bisect each other.

10. Draw a quadrilateral $ABCD$ with opposite sides AB and DC parallel. Let M be the point of intersection of the diagonals AC and BD. Through M draw a line parallel to AB that intersects AD in P and BC in Q. Prove that M is the midpoint of PQ.

11. Prove that the bisector of the apex angle of an isosceles triangle is perpendicular to the base.

12. Consider the two lines with equations $\frac{x+8}{1} = \frac{y+4}{3} = \frac{z-2}{1}$ and $(x, y, z) = (3, 3, 3) + t(4, -1, -1)$.
   a. Show that the lines are perpendicular.
   b. Find the point of intersection of the lines.

13. Determine whether the point $O(0, 0, 0)$ lies on the plane that passes through the three points $P(1, -1, 3)$, $Q(-1, -2, 5)$, and $R(-5, -1, 1)$.

14. Determine the equation in the form $Ax + By + Cz + D = 0$ of the plane that passes through the point $P(6, -1, 1)$, has z-intercept -4, and is parallel to the line $\frac{x+2}{3} = \frac{y+1}{3} = \frac{z}{-1}$.

15. Determine a point A on the line with equation $(x, y, z) = (-3, 4, 3) + t(-1, 1, 0)$, and a point B on the line $(x, y, z) = (3, 6, -3) + s(1, 2, -2)$, so that $\overrightarrow{AB}$ is parallel to $\vec{m} = (2, -1, 3)$.

16. The equation $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 9$ defines a sphere in three-dimensional space. Find the equation (in the form $Ax + By + Cz + D = 0$) of the plane that is tangent to the sphere at $(2, 4, 5)$, a point at one end of a diameter of the sphere.

17. Determine the intersection of the line $x = -1 + t$, $y = 3 + 2t$, $z = -t$ with each of the following planes:
   a. $x - y - z + 2 = 0$
   b. $-4x + y - 2z - 7 = 0$
   c. $x + 4y - 3z + 7 = 0$

18. Find the point on the xy-plane that lies on the line of intersection of the planes with equations $4x - 2y - z = 7$ and $x + 2y + 3z = 3$.

19. A plane passes through the points $(2, 0, 2)$, $(2, 1, 1)$, and $(2, 2, 4)$. A line passes through the points $(3, 2, 1)$ and $(1, 3, 4)$. Find the point of intersection of the plane and the line.

20. a. Determine the parametric equations of the line of intersection of the two planes $3x - y + 4z + 6 = 0$ and $x + 2y - z - 5 = 0$.
   b. At what points does the line of intersection intersect the three coordinate planes?
   c. Determine the distance between the xy-intercept and the xz-intercept.

21. The point Q is the reflection of $P(-7, -3, 0)$ in the plane with equation $3x - y + z = 12$. Determine the coordinates of Q.

22. Determine the components of a vector of length 44 that lies on the line of intersection of the planes with equations $3x - 4y + 9z = 0$ and $2y - 9z = 0$.

23. The line through a point $P(a, 0, a)$ with direction vector $(-1, 2, -1)$ intersects the plane $3x + 5y + 2z = 0$ at point Q. The line through P with direction vector $(-3, 2, -1)$ intersects the plane at point R. For what choice of a is the distance between Q and R equal to 3?

24. Consider two lines

   L_1: $(x, y, z) = (2, 0, 0) + t(1, 2, -1)$

   L_2: $(x, y, z) = (3, 2, 3) + s(a, b, 1)$

   where s and t are real numbers. Find a relationship between a and b (independent of s and t) that ensures that L_1 and L_2 intersect.

25. Determine all values of x, y, and z satisfying the following system of equations.

$$x + 2y - 3z = 1$$
$$2x + 5y + 4z = 1$$
$$3x + 6y - z = 3$$